# Irreducibility and Compositeness in $q$-Deformed Harmonic Oscillator Algebras

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A study of the reducibility of the Fock space representation of the $q$-deformed harmonic oscillator algebra for real and root of unity values of the deformation parameter is carried out by using the properties of the Gauss polynomials. When the deformation parameter is a root of unity, an interesting result comes out in the form of a reducibility scheme for the space representation which is based on the classification of the primitive or nonprimitive character of the deformation parameter. An application is carried out for a $q$-deformed harmonic oscillator Hamiltonian, to which the reducibility scheme is explicitly applied.

**KEY WORDS:** $q$-deformed algebras; $q$-deformed harmonic oscillator; deformation at roots of unity.

## 1. INTRODUCTION

In the last decades $q$-deformed algebras (Drinfeld, 1987; Jimbo, 1986; Kulish and Reshetikhin, 1983; Sklyanin, 1982) have been object of interest in the literature and a great effort has been devoted to its understanding and development (Biedenharn and Lohe, 1995; Chaichian and Demichev, 1996; Chari and Pressley, 1995). In particular, the interest in $q$-deformed algebras resides in the fact that they are deformed versions of the standard Lie algebras, and give them back as the deformation parameter $q$ goes to unity. Furthermore, since it is known that the deformed algebras encompass a set of symmetries that is richer than that of the Lie algebras, one is tempted to recognize that deformed algebras can be the appropriate tool to be dealt with in describing symmetries of physical systems

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which cannot be properly treated within the Lie algebras, although the direct interpretation of the deformation in these cases is sometimes incomplete or even completely lacking. On the other hand, in some cases like the XXZ-model, where the ferromagnetic/antiferromagnetic nature of a spin $\frac{1}{2}$ chain of length $N$ can be simulated through the introduction of a $q$-deformed algebra (Pasquier and Saleur, 1990), or the rotational bands in deformed nuclei and molecules which can be fitted via a $q$-rotor Hamiltonian (Celeghini *et al.*, 1992; Iwao, 1990; Raychev *et al.*, 1990), instead of using the variable moment of inertia (VMI model), the physical meaning of the deformation parameter is established. Notwithstanding this interpretation difficulty, from the original studies which appeared in connection with problems related to solvable statistical mechanics models (Baxter, 1982) and quantum inverse scattering theory (Sklyanin *et al.*, 1979), a solid development has emerged which encompass nowadays various branches of mathematical problems related to physical applications, such as deformed superalgebras (Chaichian and Kulish, 1990), knot theories (Kauffman, 1990), noncommutative geometries (Connes 1990; Manin 1988), many-body systems (Galetti *et al.*, 1995), phase transition (Avancini *et al.*, 1995; Galetti *et al.*, 1997; Galetti and Pimentel 1995) and so on. The introduction of a $q$-deformed bosonic harmonic oscillator is a subject of great interest in this context and, as a tool for providing a boson realisation of the quantum algebra $su_q(2)$, brought to light new commutation relations (Biedenharn, 1989; MacFarlane, 1989) which have been extensively discussed in the literature.

On the other hand, some concepts directly related to the arithmetical foundations of deformed algebras were well known to mathematicians since a long time (Andrews, 1975; Dickson, 1952; Gauss, 1866, 1863). For instance, the Gauss polynomials appearing in restricted partition theory (Andrews, 1975) can be directly interpreted as a $q$-generalization of the standard binomials; as such, the Gauss polynomials, or the $q$-binomials, as they are sometimes known, also generalize the concept of number as well. In that form, the Gauss extension of the number concept, sometimes known as $q$-number (Andrews, 1986; Gasper and Rahman, 1990), is also related to the usual $q$-bracket of extensive use in deformed algebras. In this connection, if, in general, the surprising effectiveness of number theory seems not to be completely realized, the success of recent examples pervading several areas can be credited to the use of that branch of science: solvable models in statistical mechanics benefited from Rogers–Ramanujan–Baxter relations, computation and cryptography, the fourth test on general relativity, dynamical systems, and primitive-roots-of-unity-based reflecting gratings in concert halls have their very foundations on basic number theory and algorithms (Stefan, 1992).

In this paper we want to address the question whether the extension of the number concept proposed by Gauss, namely the $q$-numbers, can farther help us in the study of the $q$-deformed harmonic oscillator. To this aim we are directly guided by the central role played by the number concept in this context (Galetti *et al.*, 1998). Based on this, we introduce the $Q$-numbers as our starting point to define

the action of the creation/annihilation operators on the Fock space states. From this we show how we obtain a version of the $q$-deformed harmonic oscillator algebra already discussed in the literature (Arik and Coon, 1976; Atakishiyev, 1996; Daoud *et al.*, 1998) in other contexts, namely, the $A_q$ algebra. We also discuss how a second set of operators obeying the $q$-deformed harmonic oscillator algebra can be introduced, the $\bar{A}_q$ algebra, such that they satisfy the conjugate relations with respect to the $A_q$ algebra, and discuss some possible reductions of the algebra when we choose the allowed values of the deformation parameter $q$. The cases for real and roots of unity values of $q$ are analysed. Furthermore, we also show how the reducibility of the algebra space representation appears for the different values of $q$.

Using the algebras $A_q$ and $\bar{A}_q$ we introduce a self-adjoint $q$-deformed harmonic oscillator Hamiltonian, akin to that proposed by Floratos and Tomaras (Floratos and Tomaras, 1990) and related to a system of two anyons, which allows us to test the reducibility criteria discussed before. This allows us to separate the physical systems according to the different algebras obtained for the different values of the deformation parameter $q$. In this form, we show how it is possible to distinguish different subsystems within the original oscillator Hamiltonian when $q$ assume nonprimitive roots of unity values. In this sense, we discuss the possibility of uncovering the compositeness character of the so-called $k$-fermions when discussing the reducibility of the representation space for $q$-deformed oscillator algebra at the roots of unity.

This paper is organized as follows: Section 2 is devoted to a brief review of the Gauss polynomials ($Q$-numbers). In Section 3 we derive and discuss the $q$-oscillator algebras from the extended number concept and in Section 4 we present the conditions for the reducibility of the Fock space representation. $q$-Deformed oscillator Hamiltonians are discussed in Section 5, where examples of how the reducibility conditions sieve the space representation into subspaces are also exhibited. Finally the conclusions are presented in Section 6.

## 2. GAUSS POLYNOMIALS: $Q$-NUMBERS

The generating function of restricted partitions of a positive integer $N$ into at most $m$ parts, each $\leq n$, is written as

$$G(n, m; q) = \frac{(1 - q^{n+m})(1 - q^{n+m-1}) \cdots (1 - q^{m+1})}{(1 - q)(1 - q^2) \cdots (1 - q^n)}, \tag{1}$$

$q \neq 1$, and the Gauss polynomials are defined through the relation

$$\begin{bmatrix} n \\ m \end{bmatrix} = G(n - m, m; q), \tag{2}$$

which is valid for $0 \leq m \leq n$, and zero otherwise (Andrews, 1975). The Gauss polynomial is a polynomial of degree $m(n - m)$ in $q$ that presents a very important property, namely

$$\lim_{q \to 1} \begin{bmatrix} n \\ m \end{bmatrix} = \binom{n}{m}, \tag{3}$$

where $\binom{n}{m}$ is the standard binomial. Thus, we conclude that the Gauss polynomials generalize the concept of binomials and, furthermore, as a special and important case, with $m = 1$, the Gauss polynomial, that is now denoted $Q$-number, extends the concept of number since

$$\lim_{q \to 1} \begin{bmatrix} n \\ 1 \end{bmatrix} = \binom{n}{1} = n. \tag{4}$$

On the other hand, this polynomial also allows us to establish inner contact with some aspects of number theory, since when $q$ is a $n$th root of unity, we have

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q} = 0. \tag{5}$$

This is the fundamental equation whose $n$ solutions are roots of unity; furthermore, for $n$ prime, $n - 1$ of these are primitive roots (Mathews).

Among the several relations satisfied by these polynomials a useful one, to be used later, is worth of mention (Andrews, 1975)

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n - 1 \\ m - 1 \end{bmatrix} + q^m \begin{bmatrix} n - 1 \\ m \end{bmatrix}. \tag{6}$$

## 3. $Q$-OSCILLATOR ALGEBRAS

Let us consider, as our starting point, the standard Fock space generated by $\{|n\rangle\}$,

$$a|0\rangle = 0, \tag{7}$$

$$a^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle, \qquad a|n\rangle = \sqrt{n}|n - 1\rangle, \tag{8}$$

and

$$\hat{N}|n\rangle = n|n\rangle, \tag{9}$$

where the creation and annihilation operators obey the following commutation relations

$$aa^\dagger - a^\dagger a = 1; \qquad [\hat{N}, a^\dagger] = a^\dagger; \qquad [\hat{N}, a] = -a, \tag{10}$$

from which it follows that

$$\hat{N} = a^\dagger a. \tag{11}$$

Since the number concept is inherent to the Fock description, we are strongly motivated by the results of the previous section to construct a new pair of creation and annihilation operators in such a form to deal with that generalized number concept. To this aim we introduce new operators, whose matrix elements in the Fock space involve the Gauss polynomials

$$a_- |0\rangle = 0 \tag{12}$$

$$a_+ |n\rangle = \sqrt{\{n + 1\}_q} |n + 1\rangle \tag{13}$$

$$a_- |n\rangle = \sqrt{\{n\}_q} |n - 1\rangle \tag{14}$$

$$\hat{N} |n\rangle = n |n\rangle \tag{15}$$

$$[\hat{N}, a_+] = a_+ \tag{16}$$

$$[\hat{N}, a_-] = -a_-, \tag{17}$$

although $\hat{N} \neq a_+ a_-$. Here we have adopted the notation

$$\{n\}_q \equiv \begin{bmatrix} n \\ 1 \end{bmatrix}. \tag{18}$$

We can pose now the question: what is the algebra satisfied by $a_+$ and $a_-$? Since

$$a_- a_+ |n\rangle = \{n + 1\}_q |n\rangle, \tag{19}$$

$$a_+ a_- |n\rangle = \{n\}_q |n\rangle, \tag{20}$$

and considering relation (6), we conclude that

$$a_- a_+ - q a_+ a_- = 1, \tag{21}$$

that is a $q$-deformed commutation relation as already exhibited in the literature (Arik and Coon, 1976; Atakishiyev, 1996; Daoud *et al.*, 1998). That such a set of operators indeed can be given out of the nondeformed harmonic oscillator operators can be explicitly seen by the use of the Polychronakos realization (Appendix A). In (Atakishiyev, 1996) relations (16), (17), and (21) are denoted $A_q$ algebra. We can construct an $\bar{A}_q$ algebra out of the relations conjugated to those defining the $A_q$ algebra (16), (17), and (21):

$$[\hat{N}, a_+^\dagger] = -a_+^\dagger, \tag{22}$$

$$[\hat{N}, a_-^\dagger] = +a_-^\dagger, \tag{23}$$

$$a_+^\dagger a_-^\dagger - q^* a_-^\dagger a_+^\dagger = 1. \tag{24}$$

In principle, these operators act on the dual space (bra space) to the considered Fock (ket) space. However, we can infer the action of these operators onto the ket

space just by using the orthonormality of the states $|n\rangle$. It yields

$$a_-^\dagger|n\rangle = \left(\sqrt{\{n+1\}_q}\right)^*|n+1\rangle, \tag{25}$$

and

$$a_+^\dagger|n\rangle = \left(\sqrt{\{n\}_q}\right)^*|n-1\rangle. \tag{26}$$

With the above results, it is possible to examine if there is an algebra relating the creation/anihilation operators of the $A_q$ algebra and their respective Hermitian conjugates, constituents of the $\bar{A}_q$ algebra. Using the action of these operators over the ket space, it is possible to obtain the following relations:

$$a_-a_-^\dagger = |\{\hat{N}+1\}_q|, \tag{27}$$

$$a_-^\dagger a_- = |\{\hat{N}\}_q|, \tag{28}$$

and similarly

$$a_+^\dagger a_+ = |\{\hat{N}+1\}_q|, \tag{29}$$

$$a_+a_+^\dagger = |\{\hat{N}\}_q|. \tag{30}$$

Here we shall only consider cases when $q$ is real valued or a root of unity, which are the most commonly found cases in the literature.

For real $q$ it is possible to verify that

$$|\{\hat{N}\}_q| = \{\hat{N}\}_q, \tag{31}$$

which together with Eqs. (29) and (30), and the recurrence relation of the Gauss polynomials, Eq. (6), yields:

$$a_-a_-^\dagger - qa_-^\dagger a_- = 1. \tag{32}$$

Similarly

$$a_+^\dagger a_+ - qa_+a_+^\dagger = 1. \tag{33}$$

In this case (real $q$), through Eqs. (13), (26), it is possible to identify $a_-^\dagger \equiv a_+$.

When $q$ is the fundamental root of unity it can be, by its turn, verified that

$$|\{\hat{N}\}_q| = [\hat{N}]_{q^{1/2}}, \tag{34}$$

where

$$[X]_q = \frac{q^X - q^{-X}}{q - q^{-1}} \tag{35}$$

defines the $q$-bracket of $X$. Equation (34), together with Eqs. (29) and (30) yields

$$a_-a_-^\dagger - q^{\frac{1}{2}}a_-^\dagger a_- = q^{-\frac{\hat{N}}{2}}. \tag{36}$$

Similarly

$$a_+^\dagger a_+ - q^{\frac{1}{2}} a_+ a_+^\dagger = q^{-\frac{\hat{N}}{2}}. \tag{37}$$

The last two equations characterize the $q$-oscillator algebra introduced by Biedenharn and McFarlane (Biedenharn, 1989; MacFarlane, 1989).

On the other hand, when $q$ is a root of unity, except the fundamental one, Eq. (34) is no longer valid, instead

$$|\{\hat{N}\}_q| = |[\hat{N}]_{q^{1/2}}|. \tag{38}$$

Using the definition of the $q$-bracket, Eq. (35), when $q$ is a general root of unity, $q_j = \exp(\frac{2\pi i}{m} j)$, a relation between $[k]_{q_j^{\frac{1}{2}}}$ and its $m$-complementar $[m-k]_{q_j^{\frac{1}{2}}}$, can be directly obtained

$$[m-k]_{q_j^{\frac{1}{2}}} = \frac{e^{i\frac{\pi}{m}j(m-k)} - e^{-i\frac{\pi}{m}j(m-k)}}{e^{i\frac{\pi}{m}j} - e^{-i\frac{\pi}{m}j}} = (-1)^{j-1} \frac{e^{i\frac{\pi}{m}jk} - e^{-i\frac{\pi}{m}jk}}{e^{i\frac{\pi}{m}j} - e^{-i\frac{\pi}{m}j}}$$

$$[m-k]_{q_j^{\frac{1}{2}}} = (-1)^{j-1}[k]_{q_j^{\frac{1}{2}}}. \tag{39}$$

When $q$ is the fundamental root of unity then $j = 1$, and we have

$$[m-k]_{q_1^{\frac{1}{2}}} = [k]_{q_1^{\frac{1}{2}}}. \tag{40}$$

Furthermore, for the case of the inverse of such root of unity, $q_j^{-1} = \exp(-\frac{2\pi i}{m} j) = \exp[\frac{2\pi i}{m}(m-j)]$, we can verify in exactly the same way that

$$[k]_{q_j^{-\frac{1}{2}}} = (-1)^{k-1}[k]_{q_j^{\frac{1}{2}}}. \tag{41}$$

Now, using Eqs. (39)–(41) we obtain the following additional relation

$$[m-k]_{q_j^{-\frac{1}{2}}} = (-1)^{m-k-1}[m-k]_{q_j^{\frac{1}{2}}}. \tag{42}$$

These relations will be shown to be useful when we deal with $q$-oscillator Hamiltonians in finite-dimensional spaces.

## 4. REDUCIBILITY OF THE FOCK REPRESENTATION

Now, considering the actions of $a_+$ and $a_-$ on the Fock representation, Eqs. (13) and (14), we will analyze its reducibility properties. The various possibilities are studied below.

### A. First case: $\{n\}_q = 0, \quad \forall n > 0$

All states of the $\{|n\rangle\}$ representation can be obtained through successive applications of $a_+$ over the vacuum. In that case, $\{|n\rangle\}$ is irreducible with respect to

the algebra $\{a_-, a_+, \hat{N}, I\}$.

**B. Second case: $\{m\}_q = 0$,     $\{n\}_q \neq 0$,   $\forall n, 0 < n < m$**

In that case

$$a_+ \mid m - 1\rangle = 0, \tag{43}$$

and also

$$a_- \mid m\rangle = 0, \tag{44}$$

From these results it follows that the subspace generated by $\{|0\rangle, |1\rangle, \ldots, |m - 1\rangle\}$ is invariant under the action of the set $\{a_-, a_+, \hat{N}, I\}$, and it is then an *irrep* of dimension $m$ of the deformed algebra. For all $q \neq 1$, i.e., deformed cases, the hypothesis $\{m\}_q = 0, \{n\}_q \neq 0, \forall n, 0 < n < m$ can be written as

$$\frac{q^m - 1}{q - 1} = 0, \qquad \frac{q^n - 1}{q - 1} = 0, \quad \forall n, 0 < n < m \tag{45}$$

$$q^m = 1, \qquad q^n = 1, \quad \forall n, 0 < n < m, \tag{46}$$

which is the definition of the primitive $m$th roots of unity. Therefore there will always be *irreps* of dimension $m$ whenever $q$ is a primitive $m$th root of unity.

**C. Third case: $\exists l, 0 < l < m$   $\{m\}_q = 0$,     $\{l\}_q = 0$**

This is the equivalent to state that $q$ is a nonprimitive root of unity. Let us also suppose that $l$ is the smallest integer satisfying the hypothesis above, i.e., it is the smallest number for which $q^l = 1$. Then

$$q^k = 1, \quad \forall k, 0 < k < 1, \tag{47}$$

and therefore the subspace generated by $\{|0\rangle, |1\rangle, \ldots, |m - 1\rangle\}$ is reducible in *irreps* of dimension $l$.

Labelling the $m - 1$ roots of unity as

$$q_j = e^{2\pi i \frac{j}{m}}, \quad j = 1, 2, \ldots, m - 1, \tag{48}$$

and for $r$ the greatest common divisor (GCD) of $m$ and $j$, i.e.,

$$j = rs$$

$$m = rl, \tag{49}$$

where $s/l$ is an irreducible fraction, then

$$q_j = e^{2\pi i \frac{s}{l}}. \tag{50}$$

In this way, the subspace generated by $\{|0\rangle, |1\rangle, \ldots, |m - 1\rangle\}$ is reducible, as we saw, in *irreps* of dimension $l$, which is the smallest value for which $q_j^l = 1$.

Then, for each $m$th root of unity labelled by $j$, the dimension of the *irreps* will be $l = m/r$, where $r$ is the GCD of $j$ and $m$, and the $m$-dimensional representation breaks into $r$ *irreps* of dimension $l = m/r$.

## 5. $q$-DEFORMED OSCILLATOR HAMILTONIAN

We can obtain a $q$-deformed Hermitian oscillator Hamiltonian from the deformed operator algebra presented above through the direct construction

$$H = \frac{1}{2}\hbar\omega(a_-a_-^\dagger + a_-^\dagger a_-) = \frac{1}{2}\hbar\omega(a_+a_+^\dagger + a_+^\dagger a_+), \tag{51}$$

that can be written, in general, as

$$H = \frac{1}{2}\hbar\omega\left(\sqrt{\{\hat{N}+1\}_q\{\hat{N}+1\}_q} + \sqrt{\{\hat{N}\}_q\{\hat{N}\}_{q^*}}\right), \tag{52}$$

which is equivalent to

$$H = \frac{1}{2}\hbar\omega(|\{\hat{N}+1\}_q| + |\{\hat{N}\}_q|). \tag{53}$$

As was discussed in the preceding sections

$$|\{\hat{N}\}_q| = \begin{cases} \{\hat{N}\}_q, & \text{for } q \text{ a real number} \\ |\{\hat{N}\}_{q^{1/2}}|, & \text{for } q \text{ a root of unity.} \end{cases} \tag{54}$$

So, for real $q$, the Hamiltonian, Eq. (51), is written as

$$H = \frac{1}{2}\hbar\omega(\{\hat{N}+1\}_q + \{\hat{N}\}_q), \tag{55}$$

and, for $q$ being a root of unity, it can be easily seen to reduce to

$$H = \frac{1}{2}\hbar\omega\left(\left|[\hat{N}+1]_{q_j^{1/2}}\right| + \left|[\hat{N}]_{q_j^{1/2}}\right|\right). \tag{56}$$

However, when $q$ is furthermore singled out as the fundamental primitive root of unity, the above expression, according to Eq. (34), reduces to

$$H = \frac{1}{2}\hbar\omega\left([\hat{N}+1]_{q_1^{1/2}} + [\hat{N}]_{q_1^{1/2}}\right), \tag{57}$$

which is the usual proposal for the $q$-deformed oscillator (Biedenharn, 1989). This last expression has the symmetry $q \to q^{-1}$, since this is a symmetry of the $q$-bracket itself.

Since the deformed oscillator Hamiltonian is written directly in terms of the $q$-brackets of the operator $\hat{N}$, the reducibility properties of the Fock representation space will appear in the spectrum of that operator as well. In this sense, the spectrum will be broken into blocks associated to subspaces of prime dimension whenever

the initial space dimension is a composite integer number and we work with the *nonprimitive roots of unity*.

As an application of what has been presented above we will discuss some simple cases. In this connection, we need not to work with all the roots of unity due to the properties presented for the $q$-brackets in the previous sections. In fact, using relations (39)–(42), we need not calculate the matrices representing the Hamiltonian for some primitive roots.

First, let us consider $m = 2$. In this case, the matrix representing the $q$-deformed oscillator Hamiltonian is directly written since we only have to work with the fundamental primitive root of unity, $q_1^{1/2} = \exp(i\frac{\pi}{2})$. Using the fact that $[2]_{q_1^{1/2}} = 0$, we get

$$H = \frac{1}{2}\hbar\omega \begin{pmatrix} [1]_{q_1^{1/2}} & 0 \\ 0 & [1]_{q_1^{1/2}} \end{pmatrix} = \frac{1}{2}\hbar\omega \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{58}$$

For $m = 3$, which is the next prime number, and also using Eq. (40), we get for the fundamental primitive root of unity, $q_1^{1/2} = \exp(i\frac{\pi}{3})$,

$$H = \frac{1}{2}\hbar\omega \begin{pmatrix} [1]_{q_1^{1/2}} & 0 & 0 \\ 0 & [1]_{q_1^{1/2}} + [2]_{q_1^{1/2}} & 0 \\ 0 & 0 & [2]_{q_1^{1/2}} \end{pmatrix} = \frac{1}{2}\hbar\omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{59}$$

If we now consider the case $m = 6$, we can verify how the matrix representing the Hamiltonian breaks into blocks, each with a prime dimension, as occurs in the representation space of the $q$-deformed algebra. To this end, let us first of all consider the Hamiltonian associated to the fundamental primitive root of unity, $q_1^{1/2} = \exp(i\frac{\pi}{6})$. In this case, using Eq. (39), we see that the matrix is also symmetric and has the form

$$H = \frac{1}{2}\hbar\omega \begin{pmatrix} 1 & & & & & 0 \\ & 1 + [2]_{q_1^{1/2}} & & & & \\ & & [2]_{q_1^{1/2}} + [3]_{q_1^{1/2}} & & & \\ & & & [3]_{q_1^{1/2}} + [2]_{q_1^{1/2}} & & \\ & 0 & & & [2]_{q_1^{1/2}} + 1 & \\ & & & & & 1 \end{pmatrix} \tag{60}$$

Now, if we consider the nonprimitive roots of $m = 6$, we see that there are three of them, namely, $q_2^{1/2} = \exp(i\frac{\pi}{6}2)$, $q_3^{1/2} = \exp(i\frac{\pi}{6}3)$, and $q_4^{1/2} = \exp(i\frac{\pi}{6}4)$, respectively. In fact, $q_2$ is the inverse of $q_4$ and $q_3$ is its own inverse. For the first

root the Hamiltonian matrix will be

$$
H = \frac{1}{2}\hbar\omega
\begin{pmatrix}
1 & & & & & \\
& 2 & & & 0 & \\
& & 1 & & & \\
& & & 1 & & \\
& 0 & & & 2 & \\
& & & & & 1
\end{pmatrix},
\tag{61}
$$

which breaks into two blocks, each one being the matrix associated to a $m = 3$ $q$-deformed oscillator. On the other hand, for the second nonprimitive root of unity, $q_3^{1/2} = \exp(i\frac{\pi}{6}3)$, we get

$$
H = \frac{1}{2}\hbar\omega
\begin{pmatrix}
1 & & & & & \\
& 1 & & & 0 & \\
& & 1 & & & \\
& & & 1 & & \\
& 0 & & & 1 & \\
& & & & & 1
\end{pmatrix}.
\tag{62}
$$

The three blocks associated to the $m = 2$ $q$-deformed oscillator are readily seen in this case. Since the Hamiltonian is given by (56), and using Eq. (41), we can conclude that the same matrices would be obtained if the inverse roots of unity were used. Therefore, for prime dimension spaces the matrices representing the deformed oscillator Hamiltonian are irreducible for any primitive root of unity. For nonprime integer space dimension and deformations at the *nonprimitive roots of unity*, the $q$-deformed oscillator represents in fact a composite system with as many irreducible constituents (diagonal blocks) as are the number of prime factors of the starting space dimension.

## 6. CONCLUSIONS

In the present paper, starting from the Gauss extension of the number concept, we have reobtained the $q$-deformed harmonic oscillator algebra discussed in (Arik and Coon, 1976; Atakishiyev, 1996) for general deformation parameter $q$. For the particular case of $q$ being a fundamental root of unity, we recover the deformed harmonic oscillator algebra satisfied by $a_{-(+)}$ and $a^{\dagger}_{-(+)}$ as introduced by Biedenharn and MacFarlane (Biedenharn, 1989; MacFarlane, 1989). On the other hand, some useful relations between the Gauss polynomials and the standard $q$-bracket have also been discussed for $q$ a root of unity.

A discussion on the dimensions of the Fock state space representation for $q$ real or a root of unity shows that they can be infinite as well as finite-dimensional

depending on $q$ being real or a root of unity, as it has been already pointed out. However, as a further conclusion, it is also shown that, for the particular case of $q$ being selected as a nonprimitive root of unity, the representation space, besides being finite-dimensional, also breaks into subspaces, the dimension of each block being clearly defined by the prime decomposition of the number characterizing the original space dimension. In this form, it is shown that the use of nonprimitive roots of unity allows one to verify the reducibility character of the Fock space representation, which, by its turn, shows that the subspaces characterized by prime dimensions play the role of fundamental blocks within the full space.

A $q$-deformed harmonic oscillator Hamiltonian was presented which allowed us to fully exploit the Fock space reducibility discussed previously. For the cases when $q$ was a primitive root of unity, the Hamiltonian matrix only exhibited the usual symmetries of the Q-numbers, while for $q$ being a nonprimitive root of unity (which will occur only when the space dimension is a composite number) the matrices reduced to submatrices along the diagonal, thus indicating that the original $q$-deformed oscillator is in fact made up of irreducible subsystems, each one of them of a prime dimension. This result strongly suggests the conclusion that the so called $k$-fermions, or quons, discussed in the context of roots of unity deformation, are not necessarily fundamental entities, but they may be, in some cases, composite systems made up of entities of more fundamental character. This characterization can be directly verified by studying the degree of reducibility of the space representation through the prime decomposition of the space dimension from which we started.

## APPENDIX A: POLYCHRONAKOS REALIZATION

We start from the fundamental relation (21) which clearly reduces to the classical oscillator algebra when $q \to 1$. We recall the Polychronakos realization (Polychronakos, 1990):

$$a_- = U_-(q, \hat{N})a$$
$$a_+ = U_+(q, \hat{N})a^\dagger, \tag{A1}$$

where $\hat{N}$ is the usual nondeformed number operator. Using Eq. (A1) in Eq. (21) we obtain:

$$F(q, \hat{N} + 1) - q F(q, \hat{N}) = 1, \tag{A2}$$

where

$$F(q, \hat{N} + 1) = U_+(q, \hat{N})U_-(q, \hat{N} - 1)\hat{N}. \tag{A3}$$

Representing Eq. (A3) on the Fock space $\{|n\rangle\}$ we get:

$$F(q, n+1) - q\,F(q, n) = 1. \tag{A4}$$

which is the recurrence relation (6) for the Gauss polynomials. We then may infer

$$a_+ a_- = g(q, \hat{N}) \tag{A5}$$

$$a_- a_+ = g(q, \hat{N}+1), \tag{A6}$$

or in terms of $F$

$$F(q, \hat{N}) = g(q, \hat{N}). \tag{A7}$$

In order to fulfil the deformed algebra, it is enough that

$$U_+(q, \hat{N}) U_-(q, \hat{N}-1)\hat{N} = g(q, \hat{N}). \tag{A8}$$

Choosing

$$U_-(q, \hat{N}-1) = \sqrt{\frac{\{\hat{N}\}_q}{\hat{N}}}, \qquad U_-(q, \hat{N}) = \sqrt{\frac{\{\hat{N}+1\}_q}{\hat{N}+1}}, \tag{A9}$$

we obtain

$$U_+(q, \hat{N}) = \sqrt{\frac{\{\hat{N}\}_q}{\hat{N}}}. \tag{A10}$$

Therefore

$$a_- = a\sqrt{\frac{\{\hat{N}\}_q}{\hat{N}}} \tag{A11}$$

$$a_+ = a^\dagger \sqrt{\frac{\{\hat{N}+1\}_q}{\hat{N}+1}}. \tag{A12}$$

This choice guarantees the unitarity ($a_+ = a_-^\dagger$) when $q$ is a real parameter. Otherwise, the representation turns out to be nonunitary.

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